On Waves in an Elastic Plate.

By Horace Lamb, F.R.S.

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The theory of waves in an infinitely long cylindrical rod was discussed by Pochhammer in 1876 in a well-known paper.* The somewhat simpler problem of two-dimensional waves in a solid bounded by parallel planes was considered by Lord Rayleigh† and by the present writer‡ in 1889. The main object in these various investigations was to verify, or to ascertain small corrections to, the ordinary theory of the vibrations of thin rods or plates, and the wave-length was accordingly assumed in the end to be great in comparison with the thickness.

It occurred to me some time ago that a further examination of the two-dimensional problem was desirable for more than one reason. In the first place, the number of cases in which the various types of vibration of a solid, none of whose dimensions is regarded as small, have been studied is so restricted that any addition to it would have some degree of interest, if merely as a contribution to elastic theory. Again, modern seismology has suggested various questions relating to waves and vibrations in an elastic stratum imagined as resting on matter of a different elasticity and density.§ These questions naturally present great mathematical difficulties, and it seemed unpromising to attempt any further discussion of them unless the comparatively simple problem which forms the subject of this paper should be found to admit of a practical solution. In itself it has, of course, no bearing on the questions referred to.

Even in this case, however, the period-equation is at first sight rather intractable, and it is only recently that a method of dealing with it (now pretty obvious) has suggested itself. The result is to give, I think, a fairly complete view of the more important modes of vibration of an infinite plate, together with indications as to the character of the higher modes, which are of less interest. I may add that the numerical work has been greatly simplified by the help of the very full and convenient Tables of hyperbolic and circular functions issued by the Smithsonian Institution.||

* 'Crelle,' vol. 81, p. 324 ; see also Love, 'Elasticity,' 1906, p. 275.

† 'Proc. Lond. Math. Soc.,' vol. 20, p. 225 ; 'Scientific Papers,' vol. 3, p. 249.

‡ See 'Proc. Lond. Math. Soc.,' vol. 21, p. 85.

§ Love, 'Some Problems of Geodynamics,' 1911, Chap. XI.

|| 'Hyperbolic Functions,' Washington, 1909. I have to thank Mr. J. E. Jones for kindly verifying and, where necessary, correcting my calculations.

1. The motion is supposed to take place in two dimensions x, y, the origin being taken in the medial plane, and the axis of y normal to this. The thickness of the plate is denoted by $2f$. The stress-strain equations are, in the usual notation,

$$\left.\begin{aligned} p_{xx} &= \lambda(\partial u/\partial x+\partial v/\partial y)+2\mu\,\partial u/\partial x, \\ p_{xy} &= \mu(\partial v/\partial x+\partial u/\partial y), \\ p_{yy} &= \lambda(\partial u/\partial x+\partial v/\partial y)+2\mu\,\partial v/\partial y. \end{aligned}\right\} \tag{1}$$

It is known that the solution of the equations of motion is of the type

$$u = \partial\phi/\partial x+\partial\psi/\partial y, \qquad v = \partial\phi/\partial y-\partial\psi/\partial x, \tag{2}$$

the functions ϕ, ψ being subject to the equations

$$\rho\frac{\partial^2\phi}{\partial t^2} = (\lambda+2\mu)\nabla^2\phi, \qquad \rho\frac{\partial^2\psi}{\partial t^2} = \mu\nabla^2\psi, \tag{3}$$

where
$$\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial y^2.$$

We now assume a time-factor $e^{i\sigma t}$ (omitted in the sequel), and write

$$h^2 = \frac{\rho\sigma^2}{\lambda+2\mu}, \qquad k^2 = \frac{\rho\sigma^2}{\mu}, \tag{4}$$

so that
$$(\nabla^2+h^2)\phi = 0, \qquad (\nabla^2+k^2)\psi = 0. \tag{5}$$

We further assume, for the present purpose, periodicity with respect to x. This is most conveniently done by means of a factor $e^{i\xi x}$, the wave-length being accordingly

$$\lambda' = 2\pi/\xi. \tag{6}$$

Writing
$$\alpha^2 = \xi^2-h^2, \qquad \beta^2 = \xi^2-k^2, \tag{7}$$

we have
$$\frac{\partial^2\phi}{\partial y^2} = \alpha^2\phi, \qquad \frac{\partial^2\psi}{\partial y^2} = \beta^2\psi. \tag{8}$$

The equations (1) now give

$$\left.\begin{aligned} \frac{p_{yy}}{\mu} &= (\xi^2+\beta^2)\phi-2i\xi\frac{\partial\psi}{\partial y}, \\ \frac{p_{xy}}{\mu} &= 2i\xi\frac{\partial\phi}{\partial y}+(\xi^2+\beta^2)\psi. \end{aligned}\right\} \tag{9}$$

Symmetrical Vibrations.

2. When the motion is symmetrical with respect to the plane $y = 0$ we assume, in accordance with (8),

$$\phi = \mathrm{A}\cosh\alpha y\, e^{i\xi x}, \qquad \psi = \mathrm{B}\sinh\beta y\, e^{i\xi x}. \tag{10}$$

This gives, for the stresses on the faces $y = \pm f$,

$$\left.\begin{aligned} p_{yy}/\mu &= \{\mathrm{A}(\xi^2+\beta^2)\cosh\alpha f-\mathrm{B}\,2i\xi\beta\cosh\beta f\}e^{i\xi x}, \\ p_{xy}/\mu &= \pm\{\mathrm{A}\,2i\xi\alpha\sinh\alpha f+\mathrm{B}(\xi^2+\beta^2)\sinh\beta f\}e^{i\xi x} \end{aligned}\right\} \tag{11}$$

Equating these to zero, and eliminating the ratio A/B, we have the period-equation*

$$\frac{\tanh \beta f}{\tanh \alpha f} = \frac{4\xi^2\alpha\beta}{(\xi^2+\beta^2)^2}. \tag{12}$$

In the most important type of *long* waves, ξf, αf, βf, are all small, and the limiting form of the equation is

$$(\xi^2+\beta^2)^2 - 4\xi^2\alpha^2 = 0, \tag{13}$$

whence

$$\frac{k^2}{\xi^2} = 4\left(1-\frac{h^2}{k^2}\right) = \frac{4(\lambda+\mu)}{\lambda+2\mu}, \tag{14}$$

in virtue of (4). Hence if V be the wave-velocity

$$V^2 = \frac{\sigma^2}{\xi^2} = \frac{4(\lambda+\mu)}{\lambda+2\mu}\frac{\mu}{\rho}. \tag{15}$$

This is in agreement with the ordinary theory, where the thickness is treated as infinitely small and the influence of lateral inertia is neglected.

For waves which are very *short*, on the other hand, as compared with the thickness $2f$, the quantities ξf, αf, βf are large, and the equation tends to the form

$$(2\xi^2-k^2)^2 - 4\xi^2\alpha\beta = 0. \tag{16}$$

This is easily recognised as the equation to determine the period of "Rayleigh waves" on the surface of an elastic solid.† It is known that if the substance be incompressible ($h = 0$, $\alpha = \xi$) the wave-velocity is

$$V = 0{\cdot}9554(\mu/\rho)^{\frac{1}{2}}, \tag{17}$$

whilst on Poisson's hypothesis of $\lambda = \mu$,

$$V = 0{\cdot}9194(\mu/\rho)^{\frac{1}{2}}. \tag{18}$$

These results will, in fact, present themselves later.

3. In virtue of the relations (4), (7), the equation (12) may be regarded as an equation to find σ when ξ is given, *i.e.* to determine the periods of the various modes corresponding to any given wave-length; but from this point of view it is difficult to handle. It is easier to determine the values of ξ corresponding to given values of the ratio β/α. This is equivalent to finding the wave-lengths corresponding to a given wave-velocity. Putting, in fact, $\beta = m\alpha$, we find from (7),

$$\xi^2 = \frac{k^2 - m^2h^2}{1-m^2}, \tag{19}$$

and therefore

$$V^2 = \frac{\sigma^2}{\xi^2} = \frac{k^2}{\xi^2}\frac{\mu}{\rho} = \frac{(\lambda+2\mu)(1-m^2)}{\lambda+2\mu-m^2\mu}\frac{\mu}{\rho}, \tag{20}$$

so that V depends upon m only.

* An equivalent equation is given by Rayleigh.

† Rayleigh, 'Proc. Lond. Math. Soc.,' vol. 17, p. 3 (1887); 'Scientific Papers,' vol. 2, p. 441.

4. For simplicity we will suppose in the first instance that the substance is incompressible, so that $\lambda = \infty$, $h = 0$, $\alpha = \xi$. Putting

$$\beta = m\xi, \qquad \xi f = \omega, \tag{21}$$

the equation (12) becomes

$$\frac{\tanh m\omega}{\tanh \omega} = \frac{4m}{(1+m^2)^2}, \tag{22}$$

whilst

$$V^2 = (1-m^2)\,\mu/\rho. \tag{23}$$

It is evident that real values of m must be less than unity. Moreover we have

$$\frac{d}{d\omega}\log\frac{\tanh m\omega}{\tanh \omega} = \frac{2}{\omega}\left(\frac{m\omega}{\sinh 2m\omega} - \frac{\omega}{\sinh 2\omega}\right), \tag{24}$$

which latter expression is easily seen to be positive so long as $m < 1$. Hence as ω increases from 0, the first member of (22) increases from m to its asymptotic value unity. There is therefore one and only one value of ω corresponding to any assigned value of m which makes the second member of (22) less than unity. And since

$$(1+m^2)^2 - 4m = (m-1)(m^3+m^2+3m-1), \tag{25}$$

the right-hand member of (22) is less than unity (for $m < 1$) only so long as $m < 0{\cdot}2956$, which is the positive root of the second factor in (25). The admissible real values of m therefore range from $0{\cdot}2956$ to 0. The former of these makes $\omega = \infty$, $\lambda' = 0$, and gives to V the value (17).

The values of ω corresponding to a series of values of m between the above limits, together with the corresponding values $(\lambda'/2f)$ of the ratio of wave-length to thickness, and the corresponding wave-velocities, are given in Table I (p. 118).

So far β has been taken to be real, or $\xi < k$. In the opposite case we may write

$$\beta_1^2 = k^2 - \xi^2, \tag{26}$$

and assume

$$\phi = A\cosh \alpha y\, e^{i\xi x}, \qquad \psi = B\sin \beta_1 y\, e^{i\xi x}. \tag{27}$$

The period-equation is then found to be

$$\frac{\tan \beta_1 f}{\tanh \alpha f} = \frac{4\xi^2\alpha\beta_1}{(\xi^2-\beta_1^2)^2}, \tag{28}$$

α being still supposed to be real.

In the case of incompressibility, writing

$$\xi f = \omega, \qquad \beta_1 = n\xi, \tag{29}$$

we have

$$\frac{\tan n\omega}{\tanh \omega} = \frac{4n}{(1-n^2)^2}, \tag{30}$$

whilst

$$V^2 = (1+n^2)\,\mu/\rho. \tag{31}$$

If as n increases from 0 we take always the lowest root of (30), we obtain a series of values of ω continuous with the roots of (22) and diminishing down to zero, when $n = \sqrt{3}$. This latter value makes $V = 2\sqrt{(\mu/\rho)}$, in agreement with the general formula (15) for long waves. Numerical results for a series of values of n ranging from 0 to $\sqrt{3}$ are included in Table I.

Table I.—Symmetrical Type. $\lambda = \infty$.

[The unit of V is $\sqrt{(\mu/\rho)}$.]

m.	n.	ω.	$\lambda'/2f$.	V.
0·2956	—	∞	0·0	0·9554
0·29	—	8·67	0·362	0·957
0·28	—	7·09	0·442	0·960
0·27	—	6·38	0·492	0·963
0·26	—	5·93	0·530	0·966
0·25	—	5·61	0·560	0·968
0·20	—	4·75	0·662	0·979
0·15	—	4·35	0·722	0·989
0·10	—	4·14	0·759	0·995
0·05	—	4·03	0·779	0·999
0·0	0·0	4·00	0·785	1·0
—	0·1	3·872	0·811	1·005
—	0·2	3·570	0·880	1·020
—	0·3	3·218	0·976	1·044
—	0·4	2·883	1·090	1·077
—	0·5	2·587	1·214	1·118
—	0·6	2·331	1·348	1·166
—	0·7	2·108	1·490	1·221
—	0·8	1·912	1·643	1·281
—	0·9	1·732	1·814	1·345
—	1·0	$\frac{1}{2}\pi$	2·0	$\sqrt{2}$
—	1·1	1·417	2·217	1·487
—	1·2	1·269	2·476	1·562
—	1·3	1·121	2·803	1·640
—	1·4	0·967	3·25	1·721
—	1·5	0·798	3·94	1·803
—	1·6	0·594	5·29	1·887
—	1·7	0·292	10·8	1·972
—	1·71	0·241	13·0	1·981
—	1·72	0·175	18·0	1·989
—	1·73	0·081	38·8	1·998
—	$\sqrt{3}$	0·0	∞	2·0

The relation between wave-length and wave-velocity for the whole series of modes investigated in this and in the preceding section is shown by the curve A on the opposite page. The unit of the horizontal scale is $\lambda'/2f$; that of the vertical scale is $V/\sqrt{(\mu\rho^{-1})}$.

5. On the present hypothesis of incompressibility there is a displacement function Ψ, analogous to the stream-function of hydrodynamics, viz., we have:—

$$u = \partial\Psi/\partial y, \qquad v = -\partial\Psi/\partial x. \tag{32}$$

The lines $\Psi =$ const. give the directions in which the particles oscillate, whilst if they are drawn for small equidistant values of the constant their

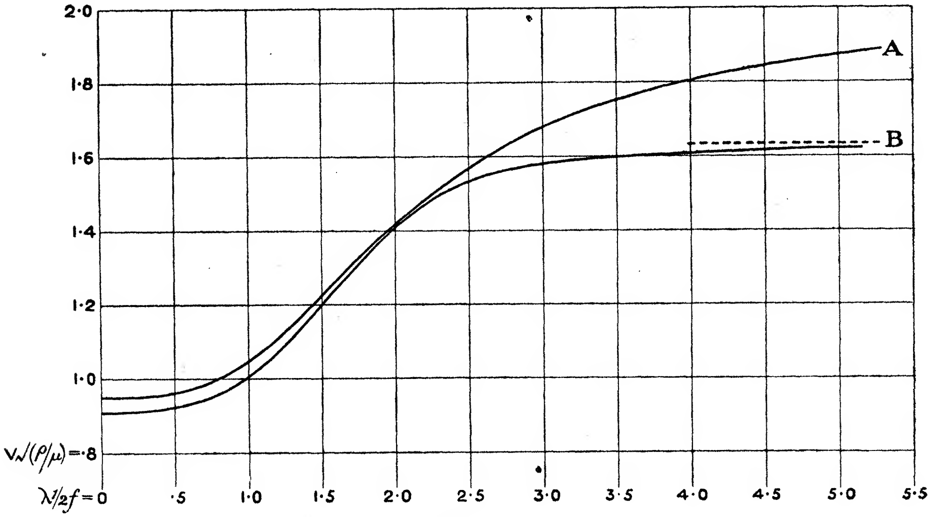

Fig. 1.

greater or less degree of closeness indicates the relative amplitudes. In the case of standing waves the system of lines retains its position in space; in the case of a progressive wave-train it must be imagined to advance with the waves.

If m be real we find, omitting a constant factor,

$$\Psi = \{2m \cosh m\omega \sinh \xi y - (1+m^2) \cosh \omega \sinh m\xi y\} e^{i\xi x}. \tag{33}$$

In the opposite case we may write

$$\Psi = \{2n \cos n\omega \sinh \xi y - (1-n^2) \cosh \omega \sin n\xi y\} e^{i\xi x}. \tag{34}$$

I have thought it worth while to make diagrams illustrating the configuration of the lines of displacement in the class of modes of vibration which have so far been obtained.

The case of the Rayleigh waves, corresponding to $\omega = \infty$, is, of course, of independent interest. In the present problem their wave-length is infinitely short; but if we transfer the origin to the surface $y = f$, and then make $f = \infty$, we get, omitting a numerical factor,

$$\Psi = (e^{m\xi y} - 0{\cdot}5437\, e^{\xi y})\, e^{i\xi x}, \tag{35}$$

where $m = 0{\cdot}2956$. On this scale the wave-length may have any value. The diagram (fig. 2) shows the great difference in the character of the motion, and the slow diminution of amplitude with increasing depth, as compared with the "surface waves" of hydrodynamics.* It follows, in fact,

* Lamb, 'Hydrodynamics,' 4th ed., art. 228.

from (35) that the ratio of the vertical amplitude to that at the surface (in the same vertical) at depths of $\frac{1}{4}$, $\frac{1}{2}$, 1 wave-length, is 1·30, 0·814, -0·340, respectively, whereas in the hydrodynamical problem the corresponding numbers are 0·208, 0·043, 0·002.

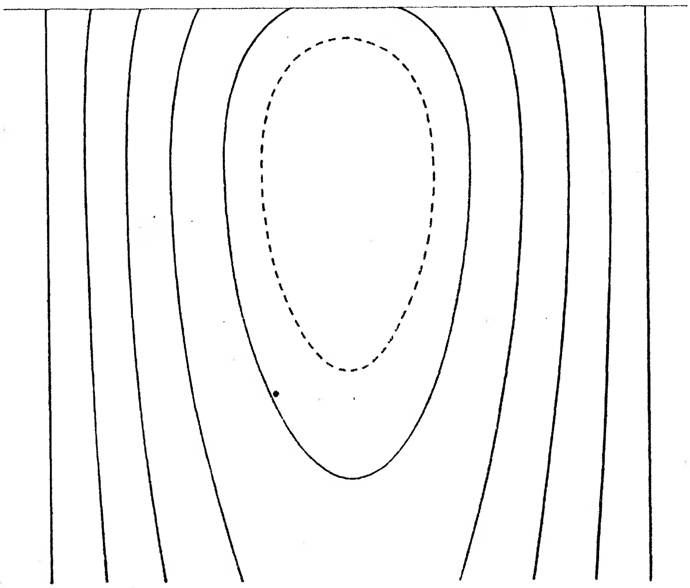

Fig. 2.

In the case of $\xi < k$, the displacement function is given by (34). I have chosen for illustration, as giving a wave-length neither too great nor too small, the case of $n = 1·6$, which makes

$$\omega = 0·594, \qquad n\omega = 0·950, \qquad \lambda'/2f = 5·29,$$

and, accordingly,

$$\Psi = (1·010 \sinh \xi y + \sin n\xi y)\, e^{i\xi x}. \tag{36}$$

The result is shown in fig. 3, which, like the former diagram, covers half a wave-length. The diagram may be taken to illustrate also, in a general

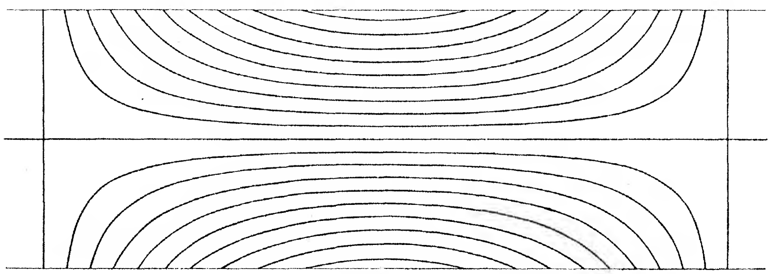

Fig. 3.

way, the most important type of longitudinal vibrations in a cylindrical rod.

6. The modes of vibration so far obtained include all the more interesting types, from the physical point of view, of the symmetrical class; but there are, of course, an infinity of others. These correspond to the higher roots of the equation (30). Thus when $n = 0{\cdot}1$ we easily find the approximate solutions

$$\omega/\tfrac{1}{2}\pi = 22{\cdot}47, \quad 42{\cdot}47, \quad 62{\cdot}47, \dots. \tag{37}$$

For $n = 0{\cdot}2$

$$\omega/\tfrac{1}{2}\pi = 12{\cdot}28, \quad 22{\cdot}28, \quad 32{\cdot}48, \dots. \tag{38}$$

For $n = 0{\cdot}3$

$$\omega/\tfrac{1}{2}\pi = 8{\cdot}72, \quad 15{\cdot}39, \quad 22{\cdot}05, \dots; \tag{39}$$

and so on.

In these modes the plane xy is mapped out into rectangular compartments whose boundaries are lines of displacement. This may be illustrated by the case of $n = 1$, when the internal compartments are squares. This happens to be particularly simple mathematically. The formula (34) for Ψ is now indeterminate, but is easily evaluated. It is found from (30) that for small concomitant variations of ω and n about $n = 1$ we have $\delta(\omega n) = 0$. This leads to

$$\Psi = \sin \xi y \, e^{i\xi x}, \tag{40}$$

with

$$\xi = (2s+1)\pi/2f, \tag{41}$$

where s is an integer. The configuration is shown in fig. 4 for the case

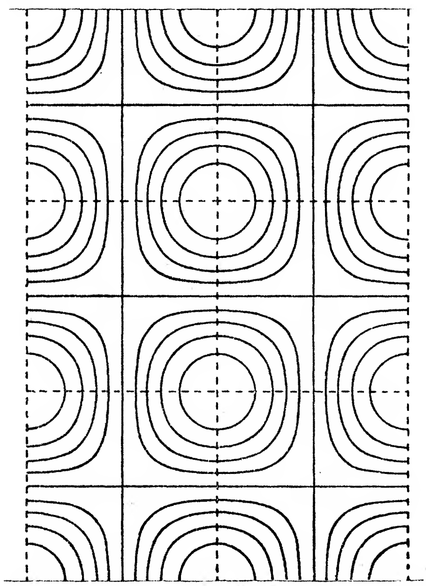

FIG. 4.

$s = 1$; but it is to be observed that the dotted lines in the diagram all represent planes which are free from stress, and that consequently any combination of them may be taken to represent free boundaries. This particular solution is, moreover, independent of the hypothesis of incompressibility.* The surface-conditions (11) are, in fact, satisfied by

$$A = 0, \qquad \xi^2 - \beta_1^2 = 0, \qquad \cos\beta_1 f = 0, \tag{42}$$

which lead again to (40) and (41).

As n is increased the compartments referred to become more elongated. For large values of n, and consequently small values of ω or ξf, we have in the limit $n\omega = s\pi$, where s is integral. It is otherwise evident that the surface conditions are satisfied by

$$A = 0, \qquad \xi = 0, \qquad \sin\beta_1 f = 0, \tag{43}$$

whence

$$\phi = 0, \qquad \psi = B\sin(s\pi y/f). \tag{44}$$

The vibration now consists of a shearing motion parallel to x, with $2s$ nodal planes symmetrically situated on opposite sides of $y = 0$. The frequency is given by

$$\sigma^2 = \frac{s^2\pi^2}{f^2}\frac{\mu}{\rho}. \tag{45}$$

Asymmetrical Modes.

7. When the motion is anti-symmetrical with respect to the plane $y=0$ we assume

$$\phi = A\sinh\alpha y\, e^{i\xi x}, \qquad \psi = B\cosh\beta y\, e^{i\xi x}, \tag{46}$$

where α, β are defined as before by (7). This gives for the stresses at the planes $y = \pm f$,

$$\left.\begin{aligned} p_{yy}/\mu &= \pm\{A(\xi^2+\beta^2)\sinh\alpha f - B\,2i\xi\beta\sinh\beta f\}e^{i\xi x}, \\ p_{xy}/\mu &= \{A\,2i\xi\alpha\cosh\alpha f + B(\xi^2+\beta^2)\cosh\beta f\}e^{i\xi x}. \end{aligned}\right\} \tag{47}$$

These surfaces being free, we deduce

$$\frac{\tanh\beta f}{\tanh\alpha f} = \frac{(\xi^2+\beta^2)^2}{4\xi^2\alpha\beta}. \tag{48}\dagger$$

* It was noticed long ago by Lamé as a possible mode of transverse vibration (uniform throughout the length) in a bar of square section, 'Théorie mathématique de l'élasticité,' 2nd ed., p. 170.

There is an analogous solution in the case of the symmetrical vibrations of a cylindrical rod. The surface-conditions given on p. 277 of Love's 'Elasticity' (equation (54)) are satisfied by

$$A = 0, \quad J_1'(\kappa' a) = 0, \quad 2\gamma^2 = p^2\rho/\mu.$$

In the notation of this paper the latter two conditions would be written

$$J_1'(\beta_1 a) = 0, \quad 2\xi^2 = k^2.$$

† *Cf.* Rayleigh, *loc. cit.*

When the waves are infinitely short this reduces to the form (16) appropriate to Rayleigh waves.

In the case of long flexural waves αf, βf, ξf are small. Writing

$$\tanh \alpha f = \alpha f(1-\tfrac{1}{3}\alpha^2 f^2), \qquad \tanh \beta f = \beta f(1-\tfrac{1}{3}\beta^2 f^2),$$

we find

$$k^2 = \tfrac{4}{3}(1-h^2/k^2)\,\xi^4 f^2, \tag{49}$$

on the supposition that k^2/ξ^2, h^2/ξ^2 are small, which is seen to be verified. This makes

$$\mathrm{V}^2 = \tfrac{4}{3}\,\xi^2 f^2 \frac{\lambda+\mu}{\lambda+2\mu}\frac{\mu}{\rho}, \tag{50}$$

in agreement with the ordinary approximate theory.

It may be pointed out in this connection that Fourier's well-known calculation* of the effect of an arbitrary initial disturbance in an infinitely long bar is physically defective, in that it rests on the assumption that the formula analogous to (50) is valid for all wave-lengths. As a result, it makes the effect of a localised disturbance begin instantaneously at all distances, whereas there is a physical limit, viz. $\sqrt{\{(\lambda+2\mu)/\rho\}}$, to the rate of propagation.

8. For the purpose of a further examination we assume the substance of the plate to be incompressible, so that $\alpha = \xi$, and write $\beta = m\xi$ as before. The equation (48) becomes

$$\frac{\tanh m\omega}{\tanh \omega} = \frac{(1+m^2)^2}{4m}, \tag{51}$$

where $\omega = \xi f$. The wave-velocity is given by (23).

Since m must be less than unity, whilst the second member of (51) exceeds 1 if $m < 0{\cdot}2956$, it appears that for real solutions we are restricted to values of m between $0{\cdot}2956$ and 1. A series of values of ω corresponding to values of m within this range is given on the next page.

The displacement-function is found to be

$$\Psi = \{(1+m^2)\cosh m\omega \cosh \xi y - 2\cosh \omega \cosh m\xi y\}\, e^{i\xi x}. \tag{52}$$

The forms of the lines $\Psi = \text{const.}$ for the case of

$$m = 0{\cdot}9, \qquad \omega = 0{\cdot}435, \qquad m\omega = 0{\cdot}392, \qquad \lambda'/2f = 7{\cdot}22,$$

are shown in fig. 5, for a range of half a wave-length. Regarded as belonging to a standing vibration, they indicate a rotation of the matter in the neighbourhood of the nodes, about these points.

* See Todhunter, 'History of the Theory of Elasticity,' vol. 1, p. 112; Rayleigh, Theory of Sound,' vol. 1, art. 192.

Table II.—Asymmetrical Type. $\lambda = \infty$.
[The unit of V is $\sqrt{(\mu/\rho)}$.]

m.	ω.	$\lambda'/2f$.	V.
0·2956	∞	0·0	0·9554
0·30	8·84	0·356	0·954
0·35	4·20	0·748	0·937
0·40	3·028	1·038	0·917
0·45	2·379	1·321	0·893
0·50	1·946	1·614	0·866
0·55	1·627	1·931	0·835
0·60	1·377	2·282	0·800
0·65	1·171	2·683	0·760
0·70	0·995	3·157	0·714
0·75	0·841	3·736	0·661
0·80	0·700	4·45	0·600
0·85	0·563	5·58	0·527
0·90	0·435	7·22	0·436
0·95	0·300	10·5	0·312
1·0	0·0	∞	0·0*

* For *small* values of ξf the value of V is given by equation (50).

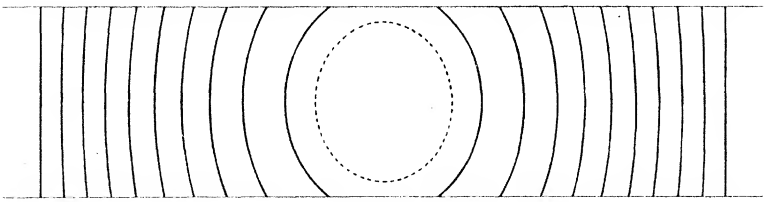

Fig. 5.

So far it has been supposed that $k < \xi$, and consequently that m is real. In the opposite case we assume in place of (46)

$$\phi = \mathrm{A} \sinh \alpha y \, e^{i\xi x}, \qquad \psi = \mathrm{B} \cos \beta_1 y \, e^{i\xi x}, \tag{53}$$

and the period-equation is

$$\frac{\tan \beta_1 f}{\tanh \alpha f} = -\frac{(\xi^2 - \beta_1^2)^2}{4\xi^2 \alpha \beta_1}, \tag{54}$$

where β_1 is defined by (26).

In the case of incompressibility this becomes

$$\frac{\tan n\omega}{\tanh \omega} = -\frac{(1-n^2)^2}{4n}, \tag{55}$$

where $\omega = \xi f$, $n = \beta_1/\xi$. As the preceding investigation (summarised in Table II) evidently covers all the modes in which v has the same sign throughout the thickness, the additional modes to which this equation relates may be dismissed with the remark that they are analogous to those referred

to in § 6 above. The particular case $n = 1$ is illustrated by fig. 4 if we imagine the lowest horizontal dotted line to form the lower boundary.

Influence of Compressibility.

9. Although the hypothesis of incompressibility has been adopted for simplicity, the numerical calculations, so far as the more important modes are concerned, are not much more complicated if we abandon this restriction. It will be sufficient to consider the case of the symmetrical types.

We have from (4) and (7)

$$\xi^2 = \frac{k^2\alpha^2 - h^2\beta^2}{k^2 - h^2} = \frac{(\lambda + 2\mu)\,\alpha^2 - \mu\beta^2}{\lambda + \mu}. \tag{56}$$

Hence if we write

$$\alpha f = \omega, \qquad \beta = m\alpha, \tag{57}$$

the equation (12) takes the form

$$\frac{\tanh m\omega}{\tanh \omega} = 4m\,\frac{(\lambda + \mu)(\lambda + 2\mu - m^2\mu)}{(\lambda + 2\mu + m^2\lambda)^2}. \tag{58}$$

The relation of ω to the wave-length is given by

$$\xi^2 f^2 = \frac{(\lambda + 2\mu - m^2\mu)\,\omega^2}{\lambda + \mu}, \tag{59}$$

whilst the wave-velocity is given by (20).

For numerical illustration, we may adopt Poisson's hypothesis as to the relation between the elastic constants. Putting, then, $\lambda = \mu$, we have

$$\frac{\tanh m\omega}{\tanh \omega} = \frac{8m(3 - m^2)}{(3 + m^2)^2}, \tag{60}$$

$$\xi f = \sqrt{\left(\tfrac{1}{2}(3 - m^2)\right)}\,\omega, \qquad \mathrm{V}^2 = \frac{3(1 - m^2)}{3 - m^2}\,\frac{\mu}{\rho}. \tag{61}$$

Real values of m must lie between 0 and 0·4641, this being the positive root of the equation

$$m^3 + 9m^2 + 15m - 9 = 0 \tag{62}$$

obtained by equating the second member of (60) to unity. The wave-velocity corresponding to this latter value of m is

$$\mathrm{V} = 0{\cdot}9194\sqrt{(\mu/\rho)}, \tag{63}$$

in accordance with the theory of Rayleigh waves, the wave-length being now infinitely small compared with the thickness.

When m is imaginary ($= in$), we have

$$\frac{\tan n\omega}{\tanh \omega} = \frac{8n(3+n^2)}{(3-n^2)^2}, \tag{64}$$

$$\xi f = \sqrt{[\tfrac{1}{2}(3+n^2)]}\,\omega, \qquad V^2 = \frac{3(1+n^2)}{3+n^2}\frac{\mu}{\rho}. \tag{65}$$

The more important modes coming under these formulæ are determined by the lowest root of (64) for values of n ranging from 0 to $\sqrt{15}$, this latter number corresponding to an infinitesimal value of ω, *i.e.* to waves of infinite length.

Numerical results are given in Table III, and the relation between wave-length and wave-velocity is shown by the curve B in fig. 1 (p. 119). The unit of the vertical scale is $V/\sqrt{(\mu\rho^{-1})}$ as before.

Table III.—Symmetrical Type. $\lambda = \mu$.
[The unit of V is $\sqrt{(\mu/\rho)}$.]

m.	n.	ω.	$\lambda'/2f$.	V.
0 ·4641	—	∞	0 ·0	0 ·9194
0 ·45	—	5 ·220	0 ·509	0 ·925
0 ·40	—	3 ·807	0 ·692	0 ·942
0 ·35	—	3 ·343	0 ·784	0 ·956
0 ·30	—	3 ·085	0 ·844	0 ·969
0 ·25	—	2 ·918	0 ·888	0 ·978
0 ·20	—	2 ·805	0 ·921	0 ·986
0 ·15	—	2 ·727	0 ·944	0 ·992
0 ·10	—	2 ·678	0 ·959	0 ·997
0 ·0	0 ·0	2 ·640	0 ·972	1 ·0
—	0 ·1	2 ·604	0 ·983	1 ·003
—	0 ·2	2 ·506	1 ·017	1 ·013
—	0 ·4	2 ·214	1 ·129	1 ·049
—	0 ·6	1 ·911	1 ·268	1 ·102
—	0 ·8	1 ·649	1 ·412	1 ·163
—	1 ·0	1 ·432	1 ·551	1 ·225
—	1 ·2	1 ·253	1 ·683	1 ·284
—	1 ·4	1 ·105	1 ·805	1 ·338
—	1 ·6	0 ·979	1 ·924	1 ·386
—	$\sqrt{3}$	0 ·907	2 ·0	$\sqrt{2}$
—	1 ·8	0 ·872	2 ·04	1 ·428
—	2 ·0	0 ·778	2 ·16	1 ·464
—	2 ·2	0 ·696	2 ·28	1 ·495
—	2 ·4	0 ·620	2 ·42	1 ·522
—	2 ·6	0 ·551	2 ·58	1 ·544
—	2 ·8	0 ·486	2 ·78	1 ·564
—	3 ·0	0 ·422	3 ·04	1 ·581
—	3 ·2	0 ·359	3 ·40	1 ·596
—	3 ·4	0 ·292	3 ·99	1 ·609
—	3 ·6	0 ·216	5 ·15	1 ·620
—	3 ·8	0 ·109	9 ·8	1 ·630
—	$\sqrt{15}$	0 ·0	∞	1 ·633

As n increases from 0, the modes corresponding to the higher roots of (64) have at first the same general character as in the case of incompressibility

(§ 6). When $n = \sqrt{3}$, we have the division into square compartments, to which fig. 4 refers. When n is infinite, whilst $n\omega$ is finite, we have

$$\alpha = 0, \qquad \beta_1 f = n\omega = s\pi, \qquad \xi f = s\pi/\sqrt{2}. \tag{66}$$

A reference to (11) shows, in fact, that, independently of any special relation between the elastic constants, the boundary conditions are satisfied by $\alpha = 0$, $\sin\beta_1 f = 0$,

and
$$\mathrm{A}(\xi^2-\beta_1^2)+2\mathrm{B}\xi\beta_1\cos s\pi = 0. \tag{67}$$

This leads to

$$\left.\begin{aligned} u &= i\left(\cos s\pi+\frac{\lambda}{2\mu}\cos\frac{s\pi y}{f}\right)e^{i\xi x}, \\ v &= \sqrt{\left\{\frac{\lambda^2}{\mu(\lambda+\mu)}\right\}}\sin\frac{s\pi y}{f}e^{i\xi x}. \end{aligned}\right\} \tag{68}$$

There is here a transition to the case where α, as well as β, is imaginary. Writing

$$\alpha_1^2 = h^2-\xi^2, \qquad \beta_1^2 = k^2-\xi^2, \tag{69}$$

and assuming (for the case of symmetry)

$$\phi = \mathrm{A}\cos\alpha_1 y\, e^{i\xi x}, \qquad \psi = \mathrm{B}\sin\beta_1 y\, e^{i\xi x}, \tag{70}$$

the period-equation is found to be

$$\frac{\tan\beta_1 f}{\tan\alpha_1 f} = -\frac{4\xi^2\alpha_1\beta_1}{(\beta_1^2-\xi^2)^2}. \tag{71}$$

Since

$$\xi^2 = \frac{h^2\beta_1^2-k^2\alpha_1^2}{k^2-h^2} = \frac{\mu\beta_1^2-(\lambda+2\mu)\alpha_1^2}{\lambda+\mu}, \tag{72}$$

we find, writing
$$\alpha_1 f = \omega, \qquad \beta_1 f = q\omega, \tag{73}$$

$$\frac{\tan q\omega}{\tan\omega} = 4(\lambda+\mu)\frac{q(\lambda+2\mu-q^2\mu)}{(q^2\lambda+\lambda+2\mu)^2}. \tag{74}$$

Also

$$\begin{aligned} \xi^2 f^2 &= \frac{q^2\mu-(\lambda+2\mu)}{(\lambda+\mu)}\omega^2, \\ \mathrm{V}^2 &= \frac{(\lambda+2\mu)(q^2-1)}{q^2\mu-(\lambda+2\mu)}\frac{\mu}{\rho}. \end{aligned} \tag{75}$$

On Poisson's hypothesis these become

$$\frac{\tan q\omega}{\tan\omega} = \frac{-8q(q^2-3)}{(q^2+3)^2}, \tag{76}$$

$$\xi f = \sqrt{[\tfrac{1}{2}(q^2-3)]}\,\omega, \qquad \mathrm{V}^2 = 3\frac{q^2-1}{q^2-3}\frac{\mu}{\rho}. \tag{77}$$

The value of q may range downwards from ∞ to $\sqrt{3}$.

A minute examination of these modes would be laborious, and would hardly repay the trouble. In the extreme case where $q = \sqrt{3}$, the equation (76) is satisfied by either a zero value of $\tan q\omega$, or an infinite value of $\tan \omega$. The former alternative gives the shearing motions parallel to x, already referred to at the end of § 6 (equations (43) and (44)). The other alternative gives a vibration at right angles to x. It is, in fact, obvious from (11) that the conditions are satisfied by

$$\xi = 0, \qquad \mathrm{B} = 0, \qquad \cos \alpha_1 f = 0, \tag{78}$$

whence

$$\phi = \cos(2s+1)\frac{\pi y}{2f}, \qquad \psi = 0, \tag{79}$$

$$\sigma^2 = (s+\tfrac{1}{2})^2 \frac{\pi^2}{f^2} \frac{\lambda + 2\mu}{\rho}. \tag{80}$$

When q slightly exceeds $\sqrt{3}$, we have modes of vibration resembling the above types, except for a gradual change of phase in the direction of x. The corresponding values of V, as given by (77), are very great, but it is to be remarked that the notion of "wave-velocity" is in reality hardly applicable (except in a purely geometrical or kinematical sense) to cases of this kind, and that results relating to modes lying outside the limits of the numerical Tables are more appropriately expressed in terms of frequency.* As already remarked, there is a physical limit to the speed of propagation of an initially local disturbance.

* *Cf.* 'Hydrodynamics,' art. 261, where a similar point arises.